Formulae (Pairs of Straight Line)

1. The homogeneous equation of second degree $ax^2 + 2hxy + by^2 = 0$, ($h^2 \geq ab$) represents a pair of straight lines passing through the origin. A second degree equation S = 0 in x, y represents a pair of lines if S is resolvable as a product of two linear factors with real coefficients.

2. a) n the equation $ax^2 + 2hxy + by^2 = 0$

i) if $h^2 > ab$, then the two lines are real and distinct and pass through the origin.

ii) if $h^2 = ab$, then the two lines are coincident.

b) If the separate equation of the pair of lines $ax^2 + 2hxy + by^2 = 0$ are $l_1 x + m_1 y = 0$

$l_2 x + m_2 y = 0$ then $l_1 l_2 = a$, $l_1 m_2 + l_2 m_1 = 2h$ and $m_1 m_2 = b$.

3. An angle θ between the pair of lines $ax^2 + 2hxy + by^2 = 0$ is given b $\cos\theta = \dfrac{a+b}{\sqrt{(a-b)^2 + 4h^2}}$ or $\tan\theta = \dfrac{2\sqrt{h^2 - ab}}{a+b}$

4. If $h^2 = ab$ then the two lines given by $ax^2 + 2hxy + by^2 = 0$ are parallel and hence coincident. Then $ax^2 + 2hxy + by^2 + 0$ is a perfect square.

5. If a + b = 0, the two lines will be perpendicular.

6. The equation of the pair of bisectors of angles between the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{x^2 - y^2}{a-b} = \dfrac{xy}{h}$ or $h(x^2 - y^2) = (a - b)xy$

7. Two pairs of lines are said to be equally inclined to each other if they possess the same pair of angle bisectors.

8. The equation to the pair of lines through the origin and perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $ax^2 - 2hxy + by^2 = 0$.

9. The equation to the pair of lines through (α, β) and perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is b

$(x - \alpha)^2 - 2h(x - \alpha)(y - \beta) + a(y - \beta)^2 = 0$.

10. The equation to pair of lines through the origin and forming an equilateral triangle with the line ax + by + c = 0 is given by $(ax + by)^2 - 3(ay - bx)^2 = 0$.

11. The area of the equilateral triangle formed by the line ax + by + c = 0 with the pair of lines $(ax + by)^2 - 3(ay - bx)^2 = 0$ is $\dfrac{c^2}{(a^2 + b^2)\sqrt{3}}$ sq. units.

12. a) The area of the triangle formed by $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is given by

$$\left| \frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2} \right|$$

b) Let $ax^2 + 2hxy + by^2 = 0$ be two sides of a parallelogram and px + qy = 1 be one diagonal. Then the equation to the other diagonal is y (bp – hq) = x (aq – hp).

13. The product of the perpendicular from (α , β) to the pair of lines $ax^2 + 2hxy + by^2$ 0 is $\dfrac{a\alpha^2 + 2h\alpha\beta + b\beta^2}{\sqrt{(a-b)^2 + 4h^2}}$

14. a) If $y = m_1 x$ and $y = m_2 x$ are the lines given by the equation $ax^2 + 2hxy + by^2 = 0$ then

$m_1 + m_2 = \dfrac{-2h}{b}$ and $m_1 m_2 = a / b$.

b) The slopes of the lines are the roots of the quadratic equation $bm^2 + 2hm + a = 0$

c) If $y = mx$ is a line of the pair $ax^2 + 2hxy + by^2 = 0$, then $bm^2 + 2hm + a = 0$.

d) The slopes of one line is λ times the other then $4\lambda h^2 = ab(1 + \lambda)^2$.

15. If (x, y) is the centroid of the triangle formed by $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$

then $\dfrac{x}{bl - hm} = \dfrac{y}{am - hl} = \dfrac{2}{3(bl^2 - 2hlm + am^2)}$

16. a) The orthocenter of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and

$lx + mx + n = 0$ is $(\lambda l, \lambda m)$ where $\lambda = \dfrac{-n(a+b)}{am^2 - 2hlm + bl^2)}$

b) The lines $ax^2 + 2hxy + by^2 = 0$ and $lx + mx + n = 0$ form an isosceles triangle is $\dfrac{l^2 - m^2}{lm} = \dfrac{a-b}{h}$.

c) If (c, d) is the orthocenter of the triangle whose two sides are the pair of lines $ax^2 + 2hxy + by^2 = 0$, then

the equation to the third side is (a + b) (cx + dy) = $(bc^2 - 2hcd + ad^2)$.

17. The equation $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines $\Rightarrow \Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $h^2 \geq ab$, $g^2 \geq ca$, $f^2 \geq bc$.

18. a) The point of intersection of the above pair of lines S = 0, when $h^2 > ab$ is $\left(\dfrac{hf - bg}{ab - h^2} , \dfrac{gh - af}{ab - h^2} \right)$.

b) If the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect at (α , β) then

(α , β) satisfy the equations $a\alpha + h\beta + g = 0$ and $g\alpha + b\beta + c = 0$.

19. The pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on x- axis, then $g^2 = ac$ & $2fgh - af^2 - ch^2 = 0$.

20 The pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on y- axis, then $f^2 = bc$ and $2fgh - bg^2 - ch^2 = 0$.

21. The angle between the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is same as the angle between the lines $ax^2 + 2hxy + by^2 = 0$.

22. If (α , β) be the point of intersection of the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then the equation to the pair of angle bisectors will be $h[(x - \alpha)^2 - (y - \beta)^2] = (a - b)[(x - \alpha)(y - \beta)]$

23. The product of the perpendiculars from a point (α , β) to the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

is $\dfrac{a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c}{\sqrt{(a-b)^2 + 4h^2}}$.

24. If p is the distance of the point of intersection of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ from the origin,

$$p^2 = \frac{c(a+b)\_- f^2 - g^2}{ab - h^2}.$$

25. The equation to the pair of lines parallel to the pair $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and passing through the origin is, $ax^2 + 2hxy + by^2 = 0$.

26. The pair of lines $ax^2 + 2hxy + by^2 = 0$ and the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ from

a) a rhombus if $(a - b) fg + h (f^2 - g^2) = 0, a + b \neq 0$ b) a square if $(a - b) fg + h (f^2 - g^2) = 0, a + b = 0$

c) a rectangle if $(a - b) fg + h (f^2 - g^2) \neq 0, a + b = 0$ d) a parallelogram if $(a - b) fg + h (f^2 - g^2) \neq 0, a + b \neq 0$

27. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel

lines if $h^2 = ab$ and $bg^2 = af^2$ or $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$. The distance between the parallel lines is $2\sqrt{\frac{g^2 - ac}{a(a+b)}}$ or $2\sqrt{\frac{f^2 - bc}{b(a+b)}}$.

28. The equation to pair of lines joining the origin to the points of intersection of the line and $lx + my = n$ and a

curve $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + c = 0$ is given by

$$Ax^2 + 2Hxy + By^2 + (2Gx + 2Fy)\left(\frac{lx+my}{n}\right) + c\left(\frac{lx+my}{n}\right)^2 = 0.$$

29.Two of the lines given by $ax^3 + bx^2y + cxy^2 + dy^3 = 0$ are at right angles if $a (a + c) + d (d + b) = 0$.

30. The condition for two pairs of lines $ax^2 + 2hxy + by^2 = 0$ and $a_1x^2 + 2hx_1y_1 + b_1y^2$

$= 0$ may have one line in common is $(ab_1 - a_1b)^2 = 4 (ah_1 - a_1h) (b_1h - bh_1)$.

31 Two of the lines represented by the equation $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0$

will be perpendicular if $(b + d) (ad + be) + (a - e)^2 (a + c + e) = 0$.

32. If (x_1, y_1) is the midpoint of the intercept made by a line L between

a) the pair of lines $S \equiv ax^2 + 2hxy + by^2 = 0$, then the equation to the line L is

$S_1 = S_{11}$ i.e. $axx_1 + h (xy_1 + x_1y) + byy_1 = ax_1^2 + 2h_1y_1 + by_1^2$

b) the pair of lines $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then the equation to the

line L is $S_1 = S_{11}$ i.e., $axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c$

$= ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$

c) If (x1, y1) is the centroid of a triangle whose two sides are given by the pair of lines

s = 0, then the equation to the third side is $S_1 \frac{3}{2}S_{11}$.